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Strain solitary waves in an elastic rod embedded in another elastic external medium with sliding

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The propagation of nonlinear elastic longitudinal strain solitary waves (i.e., strain solitons) in a cylindrical rod being in sliding contact with an external elastic medium is considered. The waves have phase velocity in an interval, determined by the elastic properties of the external medium. Furthermore, it is shown that relative to the free rod case the external medium may alter the type of strain soliton. The theory developed can formally be used to estimate the surface-tension-like effects resulting from imperfections of the rod's lateral surface. Finally, from the results obtained an approach is suggested for the possible experimental determination of the Murnaghan third-order isotropic elastic moduli. [S1063-651X(98)04309-8]

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INTRODUCTION

The study of strain *long*, quasistationary, localized waves of permanent form (solitary waves) is of theoretical and experimental interest because these waves may propagate and transfer energy over long distances along elastic waveguides. Being rather stable and powerful, the waves may cause the appearance of plasticity zones or microcracks and eventually the breakdown of a waveguide, see e.g., [1,2]. Therefore, the study of the behavior of such waves is of importance for an assessment of the durability of elastic materials and structures, methods of nondestructive testing, determination of the physical properties of both standard (e.g., brass) and new elastic materials, and, particularly, polymeric solids. Other possible applications of strain nonlinear waves come from their permanent shape property and from the dependence of their amplitude, phase velocity, etc. upon the material properties and elasticity of the waveguide.

In contrast to the nonlinear static theory the development of the nonlinear elastic *dynamic* theory is far from being complete. It is only rather recently that basic research began to appear [3–6]. The lack of understanding of nonlinear strain-wave propagation comes also from a lack of experiments. However, there are several papers devoted to both the theoretical and the experimental study of *envelope* strain solitary waves (generally, surface waves), governed, e.g., by the nonlinear Schrödinger equation (see, e.g., [7]). Only in the last decade have bulk or density strain solitary waves been studied and generated in rods [8,9] and in plates [10]. Recent experiments were motivated by the theory developed in [11]. In particular, worth noting is the generation of compression density solitons in polystyrene rods. Polystyrene absorbs well linear acoustic waves and is used in many devices [12]. Moreover, as it possesses high yield point as well as high wear and radiation resistance, it has been used as part of layered targets in nuclear fusion experiments (see [13]).

Stresses on the lateral surface of an elastic waveguide, e.g., an elastic rod, may appear due to its interaction with the surrounding external medium, as in some technological devices. Various types of contact models can be used at the interface between the rod and the external medium. The full (strong) contact model is used when there is continuity of both normal and shear stresses, and displacements. Alternatively, in a weak contact, friction may appear at the interface, hence a discontinuity in the shear stresses. Slippage provides another form of contact at the interface, in which only the continuity of the normal stresses and displacements is assumed. Surface stresses may also arise due to the imperfect manufacturing of the lateral surface of the waveguide and are formally like the ''surface tension'' on the free surface of a liquid [14,15].

The analytical solution of the contact problem is rather difficult even in the framework of the linear elasticity theory, see [16] and references therein. However, considerable progress has been achieved to account for short nonlinear surface acoustic waves propagating along the interface between elastic media [7].

Recently, in the studies of strain waves in a rod, interacting with an elastic external medium, attention was mostly focused on the propagation of *surface* strain waves along the lateral rod surface *perpendicular* to its axis (see, e.g., [17,18]). Here, however, we shall consider *bulk* density strain waves, propagating *along* the rod axis. For a recent review of the results concerning a rod with free lateral surface see [19], where the first useful approximation to tackle the problem is to reduce the three-dimensional (3D) problem to the 2D one by neglecting the rod torsion. Axial symmetry of the displacements and strain fields inside the rod is also assumed. Further simplifications can be made using explicitly some features of the physical strain *inside* the rod [20]. Thus, the so-called plane cross- sections hypothesis has been proposed for the longitudinal displacement u along the rod axis, $u(x,r,t)=U(x,t)$, while the shear displacements $w(x,r,t)$ are assumed to obey Love's relationship, $w=-\nu r U_x$ [21]. Here x and r are coordinates along the rod axis and radius, respectively; t denotes time, and ν is the Poisson ratio. Although rather useful in the study of free surface rods, these assumptions fail to properly account for contact problems, because they rule out normal stresses at the rod lateral surface, hence there is discontinuity of normal

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stresses at the interface of the rod and the external medium.

In this paper we provide an appropriate description of long nonlinear strain waves propagating in an *elastic* cylindrical rod interacting with an external and different *elastic* medium. However, we limit ourselves to the case of a *sliding* contact . The formulation of the problem is given in Sec. I. The relationships for the normal stresses acting on the lateral surface of the rod are obtained in Sec. II, by studying separately the problem inside the rod and in the surrounding elastic medium. Section III is devoted to the derivation of the relationships between strains and displacements inside the rod that satisfy the imposed boundary conditions at its lateral surface. Then an evolution equation is derived in Sec. IV for the propagation of longitudinal strain waves parallel to the axis of the rod. The influence of the elastic properties of the external medium on the solitary wave propagation inside the rod is analytically studied in Sec. V. The nonlinear temporal evolution is studied numerically in Sec. VI for the rod partly embedded in the external medium. In Sec. VII the possibility of a formal extension of the theory to account for surface tensionlike effects is discussed. A procedure is suggested for the possible determination of the Murnaghan moduli from the knowledge of the characteristics of the solitary wave propagating along the rod. Sec. VIII deals with some conclusions.

I. FORMULATION OF THE PROBLEM

Let us consider an isotropic, axially infinitely extended, elastic rod surrounded by another albeit different elastic medium, in which it may slide without friction. We shall consider the propagation of longitudinal strain waves of small but finite amplitude in the rod. Axisymmetry leads to using cylindrical Langrangian coordinates (x,r,φ), where x is the axis of the rod, $\varphi\epsilon[0,2\pi]$, $0\leqslant r\leqslant R$. When torsions are neglected, the displacement vector is $\vec{V}=(u,w,0)$. The strain field in the nonlinearly elastic medium in the reference configuration is defined by Cauchy-Green *finite* deformation tensor **C**,

$$\mathbf{C}=[\vec{\nabla}\vec{V}+(\vec{\nabla}\vec{V})^T+\vec{\nabla}\vec{V}\cdot(\vec{\nabla}\vec{V})^T]/2$$

[written in terms of a vector gradient $\vec{\nabla}\vec{V}$ and its transpose $(\vec{\nabla}\vec{V})^T$], which is the generalization of linear strain tensor. It describes the so called geometrical nonlinearity, as discussed by Engelbrecht [6]. Once the reference configuration is defined we use Hamilton's principle to obtain the evolution of nonlinear waves. Indeed, for an adiabatic deformation the Langrangian density per unit volume $\mathcal{L}$ can be obtained using the difference between the kinetic energy density K and the volume density Π of the internal energy, both per unit volume. We have

$$\mathcal{L}=K-\Pi=\frac{\rho_0}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right]-\Pi(I_k) \tag{1}$$

where ρ_0 is the rod material density at $t=t_0$, while $I_k,k=1,2,3$ are the invariants of tensor **C**:

$$I_1(\mathbf{C})=tr\mathbf{C},\quad I_2(\mathbf{C})=[(tr\mathbf{C})^2-tr\mathbf{C}^2]/2,\quad I_3(\mathbf{C})=\det\mathbf{C}. \tag{2}$$

Strains are assumed weak enough to allow neglecting any significant strain-induced thermodynamic changes. Then Π can be identified as a measure of potential strain energy density. We choose Murnaghan's approximation for deformation energy due to its applicability to a wide class of nonlinear (hyper)elastic materials [3,4]:

$$\Pi=\frac{\lambda+2\mu}{2}I_1^2-2\mu I_2+\frac{l+2m}{3}I_1^3-2mI_1I_2+nI_3. \tag{3}$$

The first two terms in Eq. (3) account for linear elasticity, hence the second order elastic moduli, or the Lamé coefficients $(\lambda,\ \mu)$, characterize linear elastic properties of the isotropic material. Other terms in Eq. (3) describe material or physical nonlinearity [3,4,6]. Accordingly, the third order elastic moduli, or the Murnaghan moduli $(l,\ m,\ n)$ (Ref. [3]) account for nonlinear elastic properties of the isotropic material. Then we set to zero the variation of the action functional,

$$\delta S=\delta\int_{t_0}^{t_1}dt\left[2\pi\int_{-\infty}^{\infty}dx\int_0^R r\mathcal{L}dr+A\right]=0, \tag{4}$$

where A is the work done by external forces. The integration in brackets in Eq. (4) is carried out at the initial time $t=t_0$. Initially, the rod is supposed to be in its natural, equilibrium state.

The displacement vector for the *linearly* elastic external medium may be written as $\vec{V}_1=(u_1,w_1,0)$. Its density is noted by ρ_1, and its elastic properties are characterized by the Lamé coefficients $(\lambda_1,\ \mu_1)$. Any disturbances due to the wave propagation inside the rod are transmitted to the external medium through displacements and stresses normal to the rod surface only when contact with *slippage* is considered. Disturbances are assumed to decay to zero in the external medium far from the rod. The normal strains as well as the displacements inside the rod are smaller than those along the rod axis. Thus we assume that displacements and strains are infinitesimal in the external medium, hence as already said it is a linear elastic one. Then for the external medium we have

$$\rho_1u_{1,tt}-(\lambda_1+2\mu_1)u_{1,zz}-(\lambda_1+\mu_1)\left(w_{1,rz}+\frac{w_{1,x}}{r}\right)$$
$$-\lambda_1\left(u_{1,rr}+\frac{u_{1,r}}{r}+w_{1,rx}+\frac{w_{1,x}}{r}\right)=0, \tag{5}$$

$$\rho_1w_{1,tt}-(\lambda_1+2\mu_1)\left(w_{1,rr}+\frac{w_{1,r}}{r}-\frac{w_1}{r^2}\right)-\mu_1w_{1,xx}$$
$$-(\lambda_1+\mu_1)u_{1,rx}=0. \tag{6}$$

The following boundary conditions (b.c.s) are imposed:

$$w\to 0,\quad \text{at } r\ \to 0, \tag{7}$$

$$w=w_1,\quad \text{at } r=R, \tag{8}$$

$$P_{rr}=\sigma_{rr},\quad \text{at } r=R, \tag{9}$$

$$P_{rx}=0, \quad \sigma_{rx}=0, \quad \text{at } r=R, \tag{10}$$

$$u_1\to 0, \quad w_1\to 0 \quad \text{at } r\to\infty, \tag{11}$$

where P_{rr}, P_{rx} denote the components of the Piola-Kirchhoff stress tensor $\mathbf{P}$ [4],

$$\begin{aligned}P_{rr}=&(\lambda+2\mu)w_r+\lambda\frac{w}{r}+\lambda u_x+\frac{\lambda+2\mu+m}{2}u_r^2\\&+\frac{3\lambda+6\mu+2l+4m}{2}w_r^2+(\lambda+2l)w_r\frac{w}{r}+\frac{\lambda+2l}{2}\frac{w^2}{r^2}\\&+(\lambda+2l)u_xw_r+(2l-2m+n)u_x\frac{w}{r}+\frac{\lambda+2l}{2}u_x^2\\&+\frac{\lambda+2\mu+m}{2}w_x^2+(\mu+m)u_rw_x,\end{aligned} \tag{12}$$

$$\begin{aligned}P_{rx}=&\mu(u_r+w_x)+(\lambda+2\mu+m)u_rw_r+(2\lambda+2m-n)u_r\frac{w}{r}\\&+(\lambda+2\mu+m)u_xu_r+\frac{2m-n}{2}w_x\frac{w}{r}+(\mu+m)w_xw_r\\&+(\mu+m)u_xw_x.\end{aligned} \tag{13}$$

The quantities σ_{rr} and σ_{rz} are the corresponding components of the linear stress tensor in the surrounding, external medium:

$$\sigma_{rr}=(\lambda_1+2\mu_1)w_{1,r}+\lambda_1\frac{w_1}{r}+\lambda_1u_{1,x} \tag{14}$$

$$\sigma_{rx}=\mu_1(u_{1,r}+w_{1,x}). \tag{15}$$

The conditions (8)–(10) define the *sliding* contact, while the longitudinal displacements u and u_1 are left free at the interface $r=R$.

The Piola-Kirchhoff tensor coincides with the linear stress tensor for infinitesimally small strains. This tensor is chosen among other finite strain tensors because it is defined in the reference configuration [4]. Note that the coefficients in P_{rr} and P_{rx} depend upon both the second-order Lamé coefficients λ and μ and the Murnaghan moduli, l,m,n. Hence the tensor $\mathbf{P}$ takes into account both the geometrical and material nonlinearities.

The linear equations (5) and (6) are solved together with the boundary conditions (8), (10), and (11), assuming that the displacement w at the interface is a given function of x and t, hence $w(x,t,R)\equiv W(x,t)$. Then the linear shear stress σ_{rr} at the interface $r=R$ is obtained as a function of W and its derivatives, thus providing the dependence only on the rod characteristics in the right-hand side of the b.c. [Eq. (9)]. The same is valid for the elementary work done by external forces at $r=R$:

$$\delta A=2\pi\int_{-\infty}^{\infty}\sigma_{rr}\delta w dx. \tag{16}$$

Satisfaction of the b.c. on the rod lateral surface yields the relationships between displacements and strains inside the rod, allowing us to separate variables in the Lagrangian (1) and to derive one nonlinear equation for long longitudinal waves using Hamilton's variational principle (4).

II. EXTERNAL STRESSES ON THE ROD LATERAL SURFACE

In this section the linear problem (5,6) will be solved with the boundary conditions (8), (10) and (11). As we focus attention on *travelling* waves along the axis of the rod we assume that all variables depend only upon the phase variable $\theta=x-ct$, where c is the phase velocity of the wave. Assuming that the unknown functions u_1,w_1 are

$$u_1=\Phi_\theta+\Psi_r+\frac{\Psi}{r}, \quad w_1=\Phi_r-\Psi_\theta, \tag{17}$$

then Φ and Ψ satisfy the equations

$$\Phi_{rr}+\frac{1}{r}\Phi_r+\left(1-\frac{c^2}{c_l^2}\right)\Phi_{\theta\theta}=0, \tag{18}$$

$$\Psi_{rr}+\frac{1}{r}\Psi_r-\frac{1}{r^2}\Psi+\left(1-\frac{c^2}{c_\tau^2}\right)\Psi_{\theta\theta}=0, \tag{19}$$

where c_l and c_τ are the velocities of the bulk longitudinal and shear linear waves in the external medium, respectively. They depend on the density and the Lamé coefficients, $c_l^2=(\lambda_1+2\mu_1)/\rho_1$, and $c_\tau^2=\mu_1/\rho_1$.

To solve Eqs. (18), (19) we introduce the Fourier transforms of Φ and Ψ:

$$\tilde{\Phi}=\int_{-\infty}^{\infty}\Phi\exp(-k\theta)d\theta, \quad \tilde{\Psi}=\int_{-\infty}^{\infty}\Psi\exp(-k\theta)d\theta$$

that reduces Eqs. (18),(19) to the Bessel equations

$$\tilde{\Phi}_{rr}+\frac{1}{r}\tilde{\Phi}_r-k^2\alpha\tilde{\Phi}=0, \tag{20}$$

$$\tilde{\Psi}_{rr}+\frac{1}{r}\tilde{\Psi}_r-\frac{1}{r^2}\tilde{\Psi}-k^2\beta\tilde{\Psi}=0, \tag{21}$$

with $\alpha=1-c^2/c_l^2$, and $\beta=1-c^2/c_\tau^2$. The ratios between c, c_l and c_τ define the signs of α and β, hence three possible sets of solutions to Eqs. (20),(21) appear, vanishing at infinity due to b.c. [Eq. (11)]. Using the boundary conditions (8),(10), we obtain the following relationships for the Fourier images of normal stresses at the lateral surface $r=R$:

(i) when $0<c<c_\tau$,

$$\begin{aligned}\tilde{\sigma}_{rr}=\frac{\mu_1\tilde{W}}{1-\beta}\Bigg(&\frac{2(\beta-1)}{R}+\frac{k(1+\beta)^2K_0(\sqrt{\alpha}kR)}{\sqrt{\alpha}K_1(\sqrt{\alpha}kR)}\\&-\frac{4k\sqrt{\beta}K_0(\sqrt{\beta}kR)}{K_1(\sqrt{\beta}kR)}\Bigg);\end{aligned} \tag{22}$$

(ii) when $c_\tau<c<c_l$,

$$\tilde{\sigma}_{rr}=\frac{\mu_1\tilde{W}}{1-\beta}\left(\frac{2(\beta-1)}{R}+\frac{k(1+\beta)^2K_0(\sqrt{\alpha}kR)}{\sqrt{\alpha}K_1(\sqrt{\alpha}kR)}-\frac{4k\sqrt{\beta}J_0(\sqrt{-\beta}kR)}{J_1(\sqrt{-\beta}kR)}\right); \tag{23}$$

(iii) when $c>c_l$

$$\tilde{\sigma}_{rr}=\frac{\mu_1\tilde{W}}{1-\beta}\left(\frac{2(\beta-1)}{R}+\frac{k(1+\beta)^2J_0(\sqrt{-\alpha}kR)}{\sqrt{-\alpha}J_1(\sqrt{-\alpha}kR)}-\frac{4k\sqrt{\beta}J_0(\sqrt{-\beta}kR)}{J_1(\sqrt{-\beta}kR)}\right); \tag{24}$$

where J_i and K_i $(i=0,1)$ denote the corresponding Bessel functions.

We shall see in the next section that in the long wave limit the normal stress σ_{rr} has one and the same functional form at the lateral surface of the rod in all three cases (22)–(24). The main difference in the stress (and strain) fields in the external medium is how they vanish at infinity. This depends on the monotonic decay of K_i and the oscillatory decay of J_i when $R\to\infty$. Note that the dependence of the strain wave behavior on the velocities of bulk linear waves, c_l, c_τ, is known, in particular, for acoustic transverse Love waves propagating in an elastic layer superimposed on an elastic half-space [6,7].

III. DERIVATION OF STRAIN-DISPLACEMENT RELATIONSHIPS INSIDE THE ROD

To solve the nonlinear problem inside the elastic rod, we have to simplify the relationships between longitudinal and shear displacements u and w. These relationships are obtained, using conditions on the free lateral surface $r=R$, namely, the simultaneous absence of the tangential stresses and the continuity of the normal ones. We search for *elastic* strain waves with sufficiently small magnitude $B\ll 1$, and a *long* wavelength relative to the rod radius R, $R/L\ll 1$. L scales the wavelength along the rod. An interesting case appears when there is balance between (weak) nonlinearity and (weak) dispersion as for a rod with free lateral surface [11,19]. Then

$$\varepsilon=B=\left(\frac{R}{L}\right)^2\ll 1 \tag{25}$$

is the smallness parameter of the problem. The linear part of longitudinal strain along the rod axis C_{xx} is u_x. Then choosing L as a scale along x, one gets BL as a scale for the displacement u. Similarly, the linear part of transverse strain, C_{rr}, is w_r. We use the scale BR for the displacement w, by choosing R as a length scale along the rod radius. Then with $|kR|\ll 1$ in (22)–(24), we have a power series expansion in kR. It allows to obtain analytically an inverse Fourier transform for σ_{rr} and to write the conditions (9),(10) in dimensionless form at the lateral surface $r=1$ as

$$(\lambda+2\mu)w_r+(\lambda-k_1)w+\lambda u_x+\frac{\lambda+2\mu+m}{2}u_r^2+\varepsilon\left(\frac{3\lambda+6\mu+2l+4m}{2}w_r^2+(\lambda+2l)ww_r+\frac{\lambda+2l}{2}w^2+(\lambda+2l)u_xw_r+(2l-2m+n)u_xw+(\mu+m)u_rw_x+\frac{\lambda+2l}{2}u_x^2-k_2w_{xx}\right)+\varepsilon^2\frac{\lambda+2\mu+m}{2}w_x^2=O(\varepsilon^3), \tag{26}$$

$$\mu u_r+\varepsilon(\mu w_x+(\lambda+2\mu+m)u_rw_r+(2\lambda+2m-n)u_rw+(\lambda+2\mu+m)u_xu_r)+\varepsilon^2\left(+\frac{2m-n}{2}ww_x+(\mu+m)w_xw_r+(\mu+m)u_xw_x\right)=O(\varepsilon^3). \tag{27}$$

At the rod lateral surface $W\equiv w$, $W_{xx}\equiv w_{xx}$. Moreover, for $0<c<c_\tau$

$$k_1=-2\mu_1,\quad k_2=\frac{\mu_1c^2(\gamma-\ln 2)}{c_\tau^2}, \tag{28}$$

while for $c_\tau<c<c_l$

$$k_1=\frac{2\mu_1(4c_\tau^2-c^2)}{c^2},$$

$$k_2=\frac{\mu_1c_\tau^2}{c^2}\left(1-\frac{c^2}{c_\tau^2}+\left(2-\frac{c^2}{c_\tau^2}\right)^2(\gamma-\ln 2)\right), \tag{29}$$

and for $c>c_l$,

$$k_1=\frac{2\mu_1[c^2(c_\tau^2-c_l^2)+3c_l^2c_\tau^2-4c_\tau^4]}{c_\tau^2(c_l^2-c^2)},\quad k_2=\frac{\mu_1c^2}{4c_\tau^2} \tag{30}$$

with $\gamma=0.577\,215\,7$ Euler's constant.

The unknown functions u,w will be found in power series of ε:

$$u=u_0+\varepsilon u_1+\varepsilon^2u_2+\cdots,\quad w=w_0+\varepsilon w_1+\varepsilon^2w_2+\cdots. \tag{31}$$

Substituting Eq. (31) in Eqs. (26), (27), and equating to zero all terms of the same order of ε, we find that the plane cross-section hypothesis and Love's relation are valid in the leading order only:

$$u_0=U(x,t),\quad w_0=rCU_x, \tag{32}$$

with

$$C=\frac{\lambda}{k_1-2(\lambda+\mu)}. \tag{33}$$

To order $O(\varepsilon)$ we get

$$u_1 = -r^2 \frac{C}{2} U_{xx}, \quad w_1 = r^3 D U_{xxx} + r Q U_x^2, \tag{34}$$

with coefficients

$$D = \frac{\lambda(\lambda + 2k_2)}{2(k_1 - 2(\lambda+\mu))(2(2\lambda+3\mu)-k_1)}, \tag{35}$$

$$Q = \frac{1}{k_1 - 2(\lambda+\mu)} \left[\frac{\lambda+2l}{2} + C(\lambda + 4l - 2m + n) + C^2(3\lambda + 3\mu + 4l + 2m) \right]. \tag{36}$$

The higher-order terms in the series (31) may be obtained in a similar way, but are omitted here being unnecessary to obtain an evolution equation for the strain waves.

IV. NONLINEAR EVOLUTION EQUATION FOR LONGITUDINAL STRAIN WAVES ALONG THE ROD AND ITS SOLUTION

Now we can derive the equation for the strain waves along the rod. First of all, substituting (31) into the potential deformation energy Π [Eq. (3], one can get in dimensionless form that

$$\Pi = a_1 U_x^2 + \varepsilon[a_2 r^2 U_x U_{xxx} + a_3 U_x^3] + O(\varepsilon^2), \tag{37}$$

with

$$a_1 = \frac{\lambda+2\mu}{2} + 2\lambda C + 2(\lambda+\mu)C^2,$$

$$a_2 = -\frac{\lambda+2\mu}{2} C - \lambda C^2 + 4\lambda D + 8(\lambda+\mu)CD,$$

$$a_3 = \frac{\lambda+2\mu}{2} + \lambda C + \lambda C^2 + 2(\lambda+\mu)C^3 + 2Q[\lambda + 2(\lambda+\mu)C] + l\left[\frac{1}{3} + 2C + 4C^2 + \frac{8}{3}C^3\right] + m\left[\frac{2}{3} - 2C^2 + \frac{4}{3}C^3\right] + nC^2.$$

For the kinetic energy we have

$$K = \frac{\rho_0}{2}[U_t^2 - \varepsilon r^2 C(U_t U_{xxt} - C U_{xt}^2)] + O(\varepsilon^2). \tag{38}$$

Substituting Eqs. (37), (38), and (16) into Eq. (4) and using Hamilton's variational principle, we obtain the following equation for a longitudinal strain wave, $v = U_x$:

$$v_{tt} - b_1 v_{xx} - \varepsilon(b_2 v_{xxtt} + b_3 v_{xxxx} + b_4 (v^2)_{xx}) = 0, \tag{39}$$

with

$$b_1 = \frac{2(a_1 - k_1 C^2)}{\rho_0}, \quad b_2 = \frac{C(1+C)}{2},$$

$$b_3 = \frac{a_2 - 2C(k_2 C + 2k_1 D)}{\rho_0}, \quad b_4 = \frac{3(a_3 - k_1 C Q)}{\rho_0}. \tag{40}$$

Equation (39) has a functional form similar to the equation obtained by Samsonov [11,19] for nonlinear waves in a rod with free lateral surface. It admits, in particular, a traveling solitary wave as an *exact* solution. Note that the coefficients depend now upon the wave velocity, c, due to Eqs. (28)–(30). The terms of order $O(\varepsilon^2)$ have been neglected, when deriving Eq. (39). Therefore we assume $c^2 = c_0^2 + \varepsilon c_1 + \cdots$ and consider the coefficients $b_2 - b_4$ depending on c_0 only, while the coefficient b_1 may depend also on c_1 as $b_1 = b_{10}(c_0) + \varepsilon b_{11}(c_0, c_1)$. Then the solitary wave solution has the form

$$v = A m^2 \cosh^{-2}(m\theta), \tag{41}$$

with

$$A = \frac{6(b_{10} b_2 + b_3)}{b_4}. \tag{42}$$

To leading order the phase velocity is obtained from the equation

$$c_0^2 = b_{10}(c_0), \tag{43}$$

and for the function $c_1$we get the equation

$$c_1 = b_{11} + 4k^2(b_{10} b_2 + b_3), \tag{44}$$

where the wave number k remains a free parameter.

V. INFLUENCE OF THE EXTERNAL MEDIUM ON THE PROPAGATION OF THE STRAIN SOLITON ALONG THE ROD

Let us estimate the influence of the external medium on the solitary wave propagation along the rod. First of all, we have to solve Eq. (43) for all three possible cases (28)–(30). As ε must not exceed the yield point of the elastic material (its usual value is less than 10^{-3}) we have to compare with c_l and c_τ the values obtained for c_0, rather than for c.

For the case (28), the velocity c_0 is obtained from Eq. (43) as

$$c_0^2 = \frac{(3\lambda+2\mu)\mu + \mu_1(\lambda+2\mu)}{\rho_0(\lambda+\mu+\mu_1)}. \tag{45}$$

It appears always higher than the wave velocity in a free rod.

For the model (29), Eq. (43) yields

$$c_0^4 - \frac{(3\lambda+2\mu)\mu + \mu_1(\lambda+2\mu) + 4\mu_1\rho_0 c_\tau^2}{\rho_0(\lambda+\mu+\mu_1)} c_0^2 + \frac{4\mu_1 c_\tau^2(\lambda+2\mu)}{\rho_0(\lambda+\mu+\mu_1)} = 0. \tag{46}$$

Finally, for the model (30), Eq. (43) provides

TABLE I. Phase velocities of waves in a polystyrene rod embedded in different media.

Material	$c_\tau \times 10^{-3}$ m/sec	$c_l \times 10^{-3}$ m/sec	$c_{01} \times 10^{-3}$ m/sec	$c_{02} \times 10^{-3}$ m/sec	$c_{03} \times 10^{-3}$ m/sec	Model
Quartz	3.78	6.02	2.06	2.1 or 7.15	2.13 or 5.77	I
Iron	3.23	5.85	2.08	2.1 or 6.32	2.11 or 5.15	I
Copper	2.26	4.7	2.07	2.11 or 4.33	2.12 or 3.68	I, II
Brass	2.12	4.43	2.06	2.11 or 4.02	2.12 or 3.45	I, II
Aluminium	3.08	6.26	2.05	2.11 or 5.75	2.13 or 4.97	I, II
Lead	1.09	2.41	2.01		1.83 or 2.06	

$$c_0^4 - \frac{(3\lambda+2\mu)\mu c_\tau^2 + (c_\tau^2 - c_l^2)\mu_1(\lambda+2\mu) + 4\mu_1\rho_0 c_\tau^4 + c_\tau^2 c_l^2 \rho_0(\lambda+\mu-3\mu_1)}{\rho_0(c_l^2\mu_1 - c_\tau^2(\lambda+\mu+\mu_1))} c_0^2 + \frac{c_\tau^2 c_l^2[3\mu_1(\lambda+2\mu) - \mu(3\lambda+2\mu)] - 4\mu_1 c_\tau^4(\lambda+2\mu)}{\rho_0(c_l^2\mu_1 - c_\tau^2(\lambda+\mu+\mu_1))} = 0. \tag{47}$$

Table I contains some quantitative estimates for a polystyrene rod and Table II for a lead rod, respectively, both embedded in different external media. The quantities c_{01}, c_{02} and c_{03} denote velocities calculated from Eqs. (45), (46), and (47), respectively. Comparing velocities c_{0i} relative to c_τ and c_l we can justify the applicability of cases (28)–(30). This is noted by symbols I–III, respectively, in the last column of Tables I and II. Indeed, the model (28) is better for the contact with a polystyrene rod, while no solitary wave may propagate when the external medium is lead. However, a solitary wave may propagate along a lead rod embedded in a polystyrene external medium, as it follows from Table II. Note that there exist pairs of materials, for which two or even all three models of sliding contact allow a solitary wave propagation. Thus the balance between nonlinearity and dispersion may be achieved at different phase velocities of the strain nonlinear waves. This result is of importance when generating strain solitary waves in a rod embedded in an external elastic medium.

Therefore, strain solitary waves can propagate only with velocities from the intervals around c_{0i}. Note that the solitary wave is a bulk (density) wave inside the rod and, simultaneously, it is a surface wave for the external medium. Then, an important difference appears relative to long nonlinear Rayleigh surface waves in Cartesian coordinates: in our case more than one velocity interval exists where solitary waves may propagate. The main difference between modes lies in the different rate of wave decay in the external medium, which follows from the different behavior of Bessel's functions at large values of their arguments.

Now let us consider the influence of the type of external medium on the existence of either compression or tensile longitudinal strain localized waves. Using the data from Table I to compute the value of A [Eq. (42)] for a polystyrene rod, it yields that its sign may change according to the values of the parameters of the material used for the external medium. Therefore the soliton (strain!) amplitude (41) may change its sign. The amplitude is negative for a free lateral surface rod and it remains negative if the external medium is, say, quartz, brass, copper, or iron. However, the sign changes if $c_0 = c_{02}$ and the external medium is aluminum. Therefore, one can anticipate, in particular, that for a rod embedded in aluminum an initial pulse with velocity close to c_{02} may transform only into a tensile soliton while an initial pulse with velocity close to c_{01} evolves to become a compression soliton.

Finally, let us consider the influence on the sign of c_1 [Eq. (44)]. For case I, $b_{11}=0$, hence the sign is defined by the sign of the quantity $(b_{10}b_2 + b_3)/b_4$. For polystyrene it is, generally, negative for all of the external media in Table I, while for a free lateral surface it is positive. Thus, the velocity c of a *nonlinear* wave in a rod embedded in an external medium is lower than the *linear* wave velocity c_0 while for a free surface rod nonlinear waves propagate faster than linear waves. On the other hand, the nonlinear wave velocity, c, in

TABLE II. Phase velocities of waves in a lead rod embedded in different external media.

Material	$c_\tau \times 10^{-3}$ m/sec	$c_{lDe} \times 10^{-3}$ m/sec	$c_{01} \times 10^{-3}$ m/sec	$c_{02} \times 10^{-3}$ m/sec	$c_{03} \times 10^{-3}$ m/sec	Model
Quartz	3.78	6.02	2.06	2.55 or 4.39	7.51	I,II,III
Iron	3.23	5.85	2.2	2.47 or 4.91	2.73 or 4.81	I, II
Copper	2.26	4.7	2.11			I
Brass	2.12	4.43	2.08			I
Aluminium	3.08	6.26	2.03			I
Polystyrene	1.01	2.1	1.83	0.38 or 1.81	1.84 or 2.06	II, III

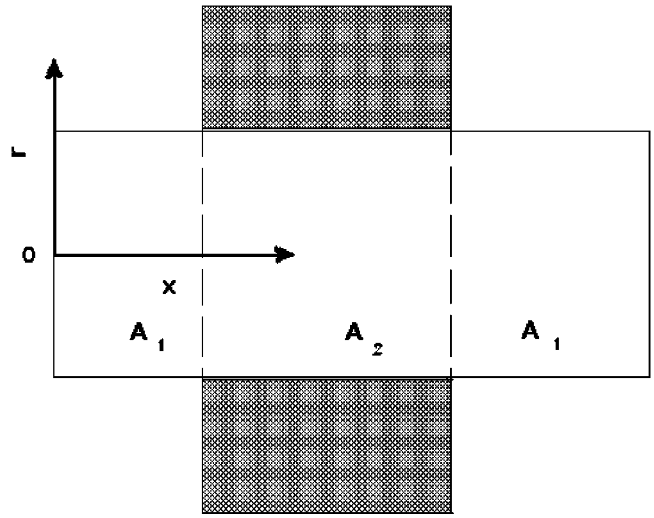

FIG. 1. Rod partly embedded in an external elastic medium with sliding.

a polystyrene rod embedded in external medium is higher than the linear wave velocity for a rod with free lateral surface, $c^* = \sqrt{E/\rho_0}$.

VI. NUMERICAL SIMULATION OF UNSTEADY STRAIN WAVE PROPAGATION

Recent numerical simulation of unsteady nonlinear wave processes in elastic rods with *free lateral surface* shows that for $A<0$ only initial compression pulses provide a solitary wave (41) or a wave train (see Fig. 3 in [19]), while tensile initial pulses do not become localized and are destroyed by dispersion. On the contrary, for $A>0$ only tensile strain solitary waves may appear, and initial compression pulses are destroyed.

Let us consider now the case when the rod lateral surface is partly free along the axis and the other part is subjected to a sliding contact with an external elastic medium, as it is shown in Fig. 1. Then the nonlinear strain wave propagation is described in each part by its own equation (39). Matching is provided by the continuity of strains and its derivatives. Assume that for the free surface part ($k_1=0$, $k_2=0$) $A=A_1$, $m=m_1$, while for the embedded one, $A=A_2$, $m=m_2$. Let the initial solitary wave (41) move from left to right (Fig. 1) far from the embedded part, which is supposed to be undeformed at the initial time. It was found in [11] that the mass M conservation in the form

$$\frac{d}{dt}M=0, \quad M=\int_{-\infty}^{\infty} v\,dx \tag{48}$$

is satisfied by equation (39). Then using Eq. (41) and (42) we get for the mass M_1,

$$M_1=2A_1m_1, \tag{49}$$

The wave evolution along the embedded part, depends on the ratio between A_1 and A_2. Similar to the unsteady processes inside a rod with the free lateral surface [19], an initial strain solitary wave will be destroyed in the embedded part, if sgn A_2 differs from sgn A_1. Otherwise another solitary wave or a wave train will appear. When the initial pulse is not massive enough it was found in [19], that only one new solitary wave appears but there is an oscillatory decaying tail. However, the contribution of the tail to the mass M is negligibly small relative to the solitary wave contribution, hence

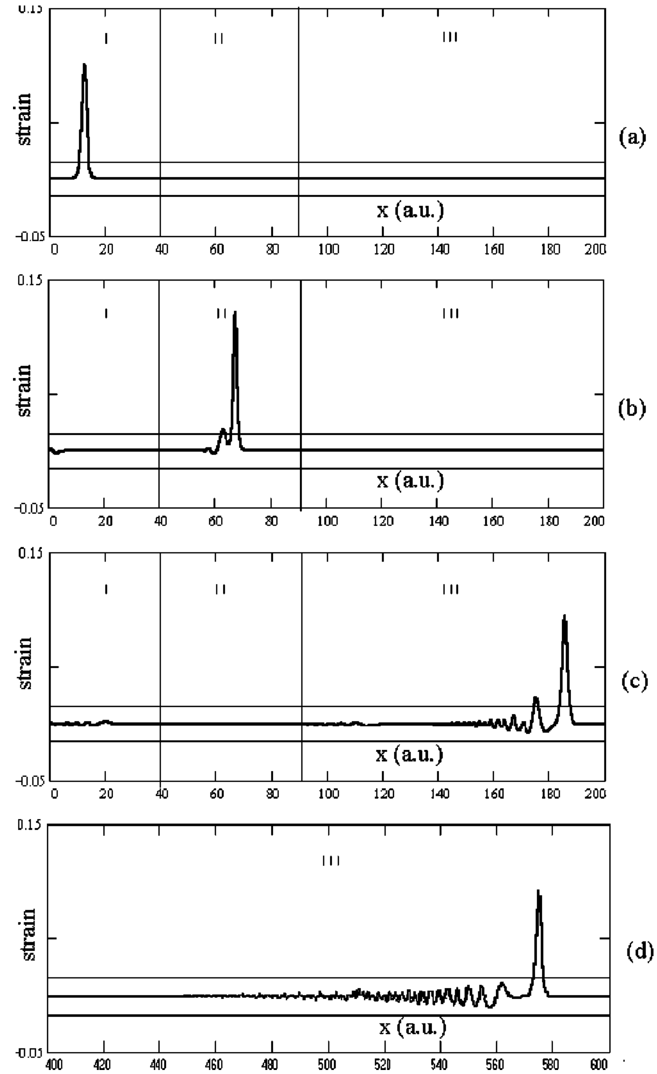

FIG. 2. Focusing and reconstruction of a strain solitary wave.

$$M_2=2A_2m_2. \tag{50}$$

Comparing M_1 and M_2, according to Eq. (48) it follows that

$$A_1m_1=A_2m_2. \tag{51}$$

Therefore, if $A_2<A_1$ the amplitude of the solitary wave increases while its width, proportional to m^{-1}, decreases, hence there is focusing of the solitary wave. On the contrary, when $A_2>A_1$ attenuation of the solitary wave is provided by the simultaneous decrease of the amplitude and the increase of the wave width.

Numerical simulations confirm our theoretical estimates. In Fig. 2 the evolution of a strain tensile solitary wave is shown in a rod, having a central part embedded in an external medium. The value of A in the central part II, A_2, is positive but smaller than the value of A_1 in the surrounding

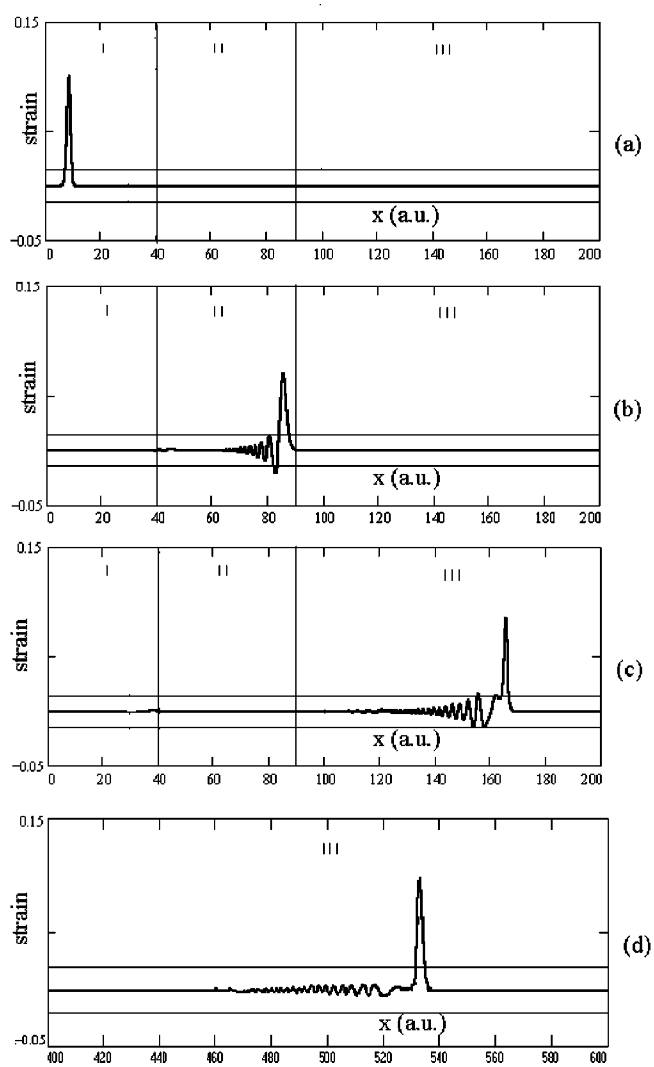

FIG. 3. Attenuation and reconstruction of a strain solitary wave.

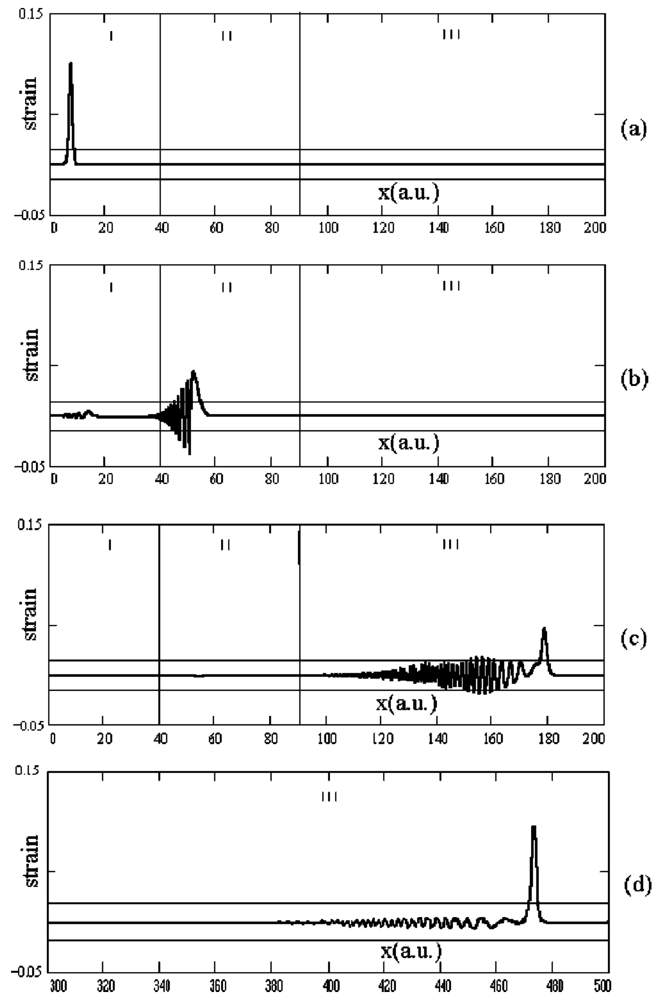

FIG. 4. Delocalization and reconstruction of a strain solitary wave.

free lateral surface parts I and III, $A_1 > A_2 > 0$. In the embedded part II [Fig. 2(b)] the solitary wave amplitude exceeds the amplitude of the initial solitary wave in Fig. 2(a), while its width becomes narrower than that of the initial wave. Therefore an increase in amplitude of the elastic strain solitary wave is possible even in an *uniformly* elastic rod. This may overtake the yield point inside the elastically deformed rod, hence the possible appearance of cracks or plasticity zones. In our case the deformations of the wave front and rear are equal. At variance with the strain soliton focusing in a rod with diminishing cross section [22] both theory and experiments show steepness of the wave front together with widening of its rear. Moreover, a *plateau* develops in the tail of the solitary wave. These differences could be caused by the absence of mass (and energy) conservation for strain solitary waves in a rod with diminishing cross section.

In the case treated in this paper, the solitary wave does not lose mass M, hence its original shape is recovered when traversing part III in Fig. 2(c,d). One can see that an oscillatory tail of the solitary wave in Fig. 2(d) is less pronounced than the tail in Fig. 2(c), in agreement with Eq. (51).

When $A_2 > A_1 > 0$, an initial tensile strain solitary wave, Fig. 3(a), is drastically attenuated as soon as it enters the embedded area, Fig. 3(b), and its amplitude decreases while its width becomes larger. Again both the reconstruction of the initial wave profile and the damping of its tail are observed in the third part of a rod with free lateral surface, part III in Fig. 3(c,d).

Consider now the case of different signs of A_i and assume that $A_1 > 0$ on both free surface parts. One can see in Fig. 4 how an initial tensile solitary wave, Fig. 4(a), is destroyed in the embedded part II, Fig. 4(b), in agreement with our previous results on the unsteady processes occurring for a free surface rod. However, a strain wave is localized again in the third part of a rod with free lateral surface, Fig. 4(c), part III, and finally recovers its initial shape in Fig. 4(d). Again damping of the tail behind the solitary wave is observed. Accordingly, both compression and tensile initial pulses may produce localized strain solitary waves in a rod partly embedded in an external elastic medium with sliding.

Moreover, the amplitude of the solitary wave generated in such a manner may be greater than the magnitude of the initial pulse. This case is shown in Figs. 5, 6 where $A_1 < 0$, $A_2 > 0$, and $|A_1| < A_2$. One can see in Fig. 5 how an initially localized rectangular tensile pulse, Fig. 5(a), is destroyed in the free surface part I, Fig. 5(b). However, a wave train of solitons appears, when a destroyed strain wave comes to the embedded part, Figs. 5(c,d). The amplitude of the first soliton in Fig. 5(d) exceeds the magnitude of the

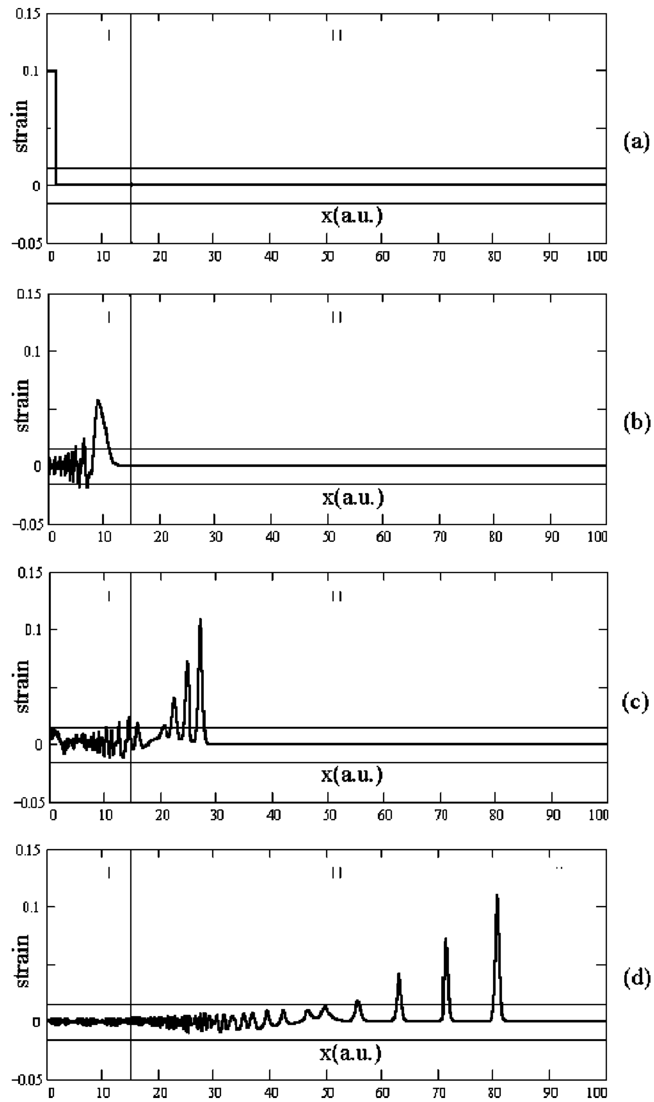

FIG. 5. Generation of a tensile strain solitary wave train in a rod. The elastic properties of the rod are chosen such that tensile wave propagation cannot occur in the absence of contact with an external medium.

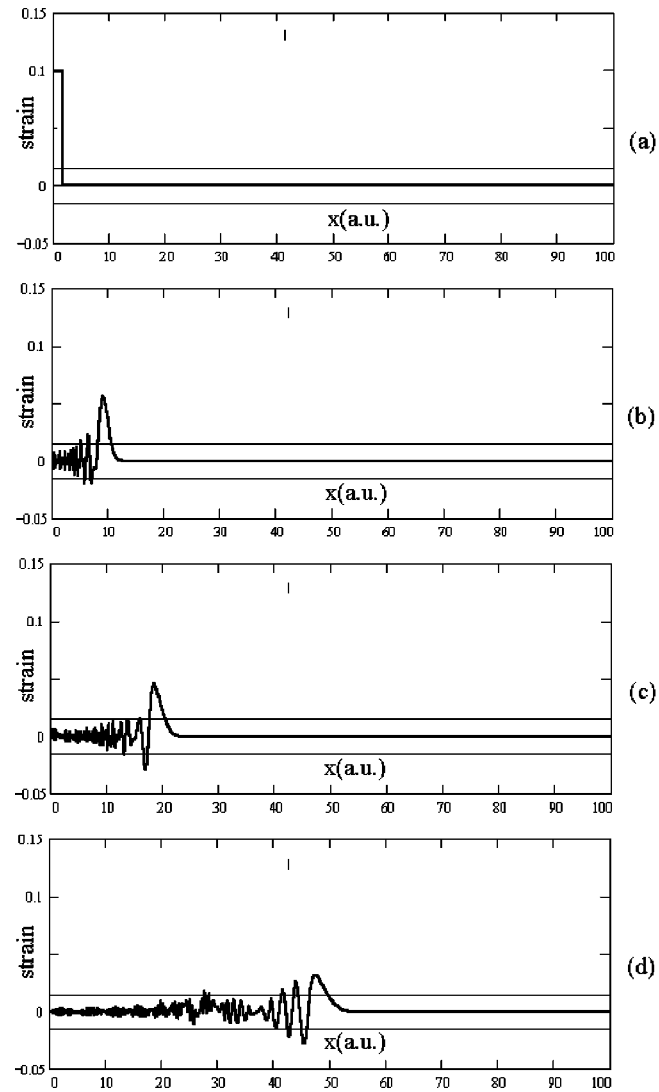

FIG. 6. Delocalization of a strain solitary wave in the absence of external medium.

initial rectangular pulse in Fig. 5(a). In the absence of surrounding external medium this rod waveguide does not support tensile solitary wave propagation, and a strain wave is delocalized as shown in Fig. 6.

VII. POSSIBLE APPLICATIONS OF THE THEORY

A. Surface-tension-like effects

The theory developed may be applied to the study of surface-tension-like effects in solids, when there are imperfections on the rod surface, see, e.g., [14]. Recently, it was experimentally found [15] that the stresses due to surface effects, may be rather large. Theory [14] shows that surface stresses, acting on the lateral surface of an elastic body, may be modelled by using normal stresses in the form

$$\sigma_{rr}=\alpha_{eff}w_{xx}, \tag{52}$$

with α_{eff} being a surface-tension-like coefficient. In this case the boundary conditions (26), (27) are valid with $k_1=0$, $k_2=\alpha_{eff}$. Thus our theory may be *formally* extended to account for the influence of surface-tension-like effects on the propagation of strain solitary waves. This ''surface tension'' does not alter the phase velocity c though it may affect the sign of the wave amplitude. Although the problem of obtaining meaningful values of the ''surface tension'' coefficient in solids is far from being solved, the data given in [15] for some materials seem to be reliable. The theory developed here may be used for the determination of the surface-tension-like coefficient. Indeed, the expression $(b_{10}b_2+b_3)/b_4$ contains α_{eff}, hence, by measuring solitary wave parameters in a rod with different surface roughness, one can obtain the corresponding values of α_{eff}. Accordingly, an estimation of the influence of the surface tension on the solitary wave parameters is useful for applying the theory to nondestructive testing, because a bulk strain solitary wave (41) keeps its shape, independently of the lateral surface roughness, while the wave parameters (amplitude, velocity, etc.) contain information about it.

B. Murnaghan's moduli

The isotropic third-order Murnaghan's moduli (l,m,n) are not known for many materials. The third-order crystalline

TABLE III. Deviations in *percent* from Eq. (53).

Material	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6
Aluminium	60	15	111	92	208	50
Molybdenum	27	52	115	305	604	39

moduli have been measured for many materials [4], [23], and it was proposed in [24] to use them to obtain the isotropic moduli. For cubic crystals the relationships are

$$c_{112}=2l,\quad c_{166}=m,\quad c_{456}=n/4,\quad c_{123}=n-2m+2l,$$
$$c_{144}=m-n/2,\quad c_{111}=4m+2l, \tag{53}$$

where c_{ijk} denotes the corresponding third-order crystalline elastic modulus for cubic crystals. However, independent measurements of isotropic moduli for some materials do not satisfy these analytical relationships. For instance, for aluminum and molybdenum, for which both the Murnaghan moduli and the crystalline cubic moduli [23] are known, we can estimate the discrepancy. Using Eqs. (53) we calculate the deviations, see Table III,

$$\delta_1=\left|\frac{c_{112}-2l}{c_{112}}\right|;\quad \delta_2=\left|\frac{c_{166}-m}{c_{166}}\right|;\quad \delta_3=\left|\frac{c_{456}-0.25n}{c_{456}}\right|;$$

$$\delta_4=\left|\frac{c_{123}-n+2m-2l}{c_{123}}\right|;\quad \delta_5=\left|\frac{c_{144}-m+0.5n}{c_{144}}\right|;$$

$$\delta_6=\left|\frac{c_{111}-4m-2l}{c_{111}}\right|.$$

Indeed, there is interest in the direct measurement of the Murnaghan moduli. Our theory gives one possible way. We see from Eq. (40) that

$$b_4=q_0+q_1l+q_2m+q_3n \tag{54}$$

is a linear combination of Murnaghan's moduli, with

$$q_0=\frac{H}{2\rho_0[2(\lambda+\mu)-k_1^2]^4},$$

where

$$H=48\mu(\lambda+\mu)^3(3\lambda+2\mu)+3k_1^4(\lambda+2\mu)-3k_1^3(\lambda+4\mu)$$
$$\times(5\lambda+4\mu)+12k_1^2(3\lambda+2\mu)(\lambda^2+6\lambda\mu+6\mu^2)$$
$$-6k_1(\lambda+\mu)(3\lambda+2\mu)(\lambda^2+16\lambda\mu+16\mu^2);$$

and

$$q_1=\frac{(2\mu-k_1)[k_1^2+k_1(\lambda-4\mu)+4\mu(\lambda+\mu)]}{\rho_0[2(\lambda+\mu)-k_1^2]^4};$$

$$q_2=\frac{2(k_1-3\lambda-2\mu)[k_1^3-k_1^2(5\lambda+6\mu)+k_1(3\lambda^2+20\lambda\mu+12\mu^2)-4\mu(3\lambda^2+5\lambda\mu+2\mu^2)]}{\rho_0[2(\lambda+\mu)-k_1^2]^4};$$

$$q_3=\frac{6\lambda^2(\lambda+\mu-k_1)}{\rho_0[2(\lambda+\mu)-k_1^2]^3}.$$

The coefficients q_0-q_4, b_4 are functions of usually known Lamé coefficients and densities of the rod and the external medium. The coefficient b_4 additionally depends upon the wave amplitude (41),(42). Hence, Eq. (54) may be considered as a linear inhomogeneous algebraic equation for the Murnaghan moduli (l,m,n). Taking three different external media we may have three equations and obtain the values of l,m,n. The necessary and sufficient condition for a nontrivial solution is the nonzero value of the determinant of the system. Calculations for several elastic materials show that it usually does not vanish.

As the width or wavelength of a solitary wave does not have a precise definition it is better to search for periodic wave trains. Equation (39), indeed, admits such a solution in the form of a cnoidal wave

$$v=Ak^2\left[1-\frac{E}{K}-\kappa^2+\kappa^2cn^2(k\theta|\kappa)\right] \tag{55}$$

where $K(\kappa),E(\kappa)$ and κ are the complete elliptic integrals of the first and second kinds and Jacobi functions modulus, respectively, A is defined by Eq. (42). For the wave number k we have

$$k=\frac{2K(\kappa)}{L_{cn}}, \tag{56}$$

where L_{cn} is the length of the cnoidal wave, which is defined as a distance between neighboring maxima or minima. Suppose, for example, $A>0$. Then

$$v_{\max}=Ak^2\left[1-\frac{E}{K}\right],\quad v_{\min}=Ak^2\left[1-\frac{E}{K}-\kappa\right]. \tag{57}$$

Using Eq. (57) the value of the Jacobi modulus κ [and, therefore both $K(\kappa)$ and $E(\kappa)$] may be obtained from the equation

$$\left[1-\frac{E}{K}\right](v_{\max}-v_{\min})-\kappa^2v_{\max}=0,$$

where $v_{\max}$, $v_{\min}$ may be measured as for the solitary wave amplitude in [8,9]. Next, the wave number may be obtained from Eq. (56) using the measured wavelength L_{cn}. Finally either $v_{\max}$ or $v_{\min}$ may be used for the procedure described for the solitary wave case. For negative A we have to exchange $v_{\max}$, $v_{\min}$ in Eq. (57). The influence of dissipation on the propagation of nonlinear elastic waves in a rod has been estimated in [8]. It was found that it cannot be important, and cause negligible wavelength variation. Moreover, weak dissipation may be described analytically as with the influence of slow variable cross-sections of the rod [22]. Therefore, with a cnoidal wave there seems to be no problem in determining with high accuracy all wave characteristics. Unfortunately, to our knowledge no experimental data are available concerning the generation of such a wave even in a rod with free lateral surface. Thus we leave this matter as a challenge for experimentalists.

VIII. CONCLUSIONS

A theory has been developed for the description of nonlinear longitudinal strain waves in an elastic rod embedded in another external elastic medium with sliding contact. First, relationships were obtained for the normal stresses acting on the rod lateral surface. Then, in the long wave limit we derived the nonlinear evolution equation for strain waves along the rod, and an exact solitary wave solution has been obtained. The analysis of the solution allowed us to conclude that the influence of the external medium defines an interval of phase velocities in which a solitary wave can propagate.

In contrast to *surface* wave propagation in Cartesian coordinates for waveguides, where only one wave velocity is possible, here we have two or even three intervals of ''allowed'' velocities. Moreover, depending on the elasticity of the surrounding external medium the longitudinal strain wave in a rod may be a tensile or a compression wave.

We have also numerically followed the evolution of the nonlinear wave in a rod partly embedded in an elastic external medium. Focusing, attenuation, or delocalization of a strain solitary wave is observed in such a case depending on the elastic properties of the external medium. Moreover, in each of these cases there is reappearance of the original solitary wave when reemerging from the embedded area. As a result of wave focusing exceeding the yield point of the elastic rod material may occur, as well as the possibility of localization of both compression or tensile pulses. All of these properties could be useful when designing elastic structures, or establishing criteria to assess their durability and fracture mechanics.

A generalization of the theory has been proposed to formally account for surface-tension-like effects on the evolution of long nonlinear strain waves. This extension of the theory may also be of interest for using nonlinear waves as probes in nondestructive testing. Finally, we have shown how the theory has potential for a direct determination of the Murnaghan third-order isotropic elastic moduli of the material by measuring the parameters of the wave propagating along the rod.

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[1] J. Coste and C. Montes, Phys. Rev. A **34**, 3940 (1986).
[2] C. Montes, Phys. Rev. A **36**, 2976 (1987).
[3] F. D. Murnaghan, *Finite Deformations of an Elastic Solid* (Wiley, New York, 1951).
[4] A. I. Lurie, *Nonlinear Theory of Elasticity* (Elsevier, Amsterdam, 1990).
[5] D. R. Bland, *Nonlinear Dynamic Elasticity* (Blaisdell, Waltham, MA, 1969).
[6] *Nonlinear Waves in Solids*, edited by A. Jeffrey and J. Engelbrecht (Springer, New York, 1994).
[7] *Recent Developments in Surface Acoustic Waves*, edited by D. F. Parker and G. A. Maugin (Springer, Berlin, 1987).
[8] G. V. Dreiden, Yu. I. Ostrovsky, A. M. Samsonov, I. V. Semenova, and E. V. Sokurinskaya, Zh. Tekh. Fiz. **58**, 2040 (1988) [Sov. Phys. Tech. Phys. **33**, 1237 (1988)].
[9] G. V. Dreiden, A. V. Porubov, A. M. Samsonov, I. V. Semenova, and E. V. Sokurinskaya, Tech. Phys. Lett. **21**, 415 (1995).
[10] A. M. Samsonov, G. V. Dreiden, A. V. Porubov, and I. V. Semenova, Tech. Phys. Lett. **22**, 891 (1996).
[11] A. M. Samsonov, Dokl. Akad. Nauk. **299**, 1083 (1988) [Sov. Phys. Dokl. **33**, 298 (1988)]; **277**, 332 (1984) [**29**, 586 (1984)].
[12] V. A. Shutilov, *Foundations of the Physics of Ultrasound* (University Press, Leningrad, 1980) (in Russian).
[13] A. V. Bushman, I. V. Lomonosov, V. I. Fortov, K. V. Khishchenkol, M. V. Zhernokletov, and Yu. N. Sutulov, JETP **82**, 895 (1996).
[14] S. V. Biryukov, Yu. V. Gulyaev, V. V. Krylov, and V. P. Plessky, *Surface Acoustic Waves in Inhomogeneous Media* (Nauka, Moscow, 1991) (in Russian).
[15] E. G. Nikolova, Zh. Éksp. Teor. Fiz. **72**, 545 (1977) [Sov. Phys. JETP **45**, 285 (1977)].
[16] A. D. Kerr, J. Appl. Mech. **31**, 491 (1964).
[17] N. S. Shevyakhov, Akust. Zh. **23**, 155 (1977) [Sov. Phys. Acoust. **23**, 86 (1977)].
[18] Yu. V. Gulyaev and N. I. Polsikova, Akust. Zh. **24**, 504 (1978) [Sov. Phys. Acoust. **24**, 287 (1978)].
[19] A. M. Samsonov, Acta Tech. CSAV **42**, 93 (1997).
[20] H. D. McNiven and J. J. McCoy, in *R. Mindlin and Applied Mechanics*, edited by G. Herrmann (Pergamon, New York, 1974), pp. 197–226.
[21] A.E.H. Love, *A Treatise on the Mathematical Theory of Elasticity* (University Press, Cambridge, 1927).
[22] A. M. Samsonov, G. V. Dreiden, A. V. Porubov, and I. V. Semenova, Phys. Rev. B **57**, 5778 (1998).
[23] I. N. Frantsevich, F. F. Voronov, and S. A. Bakuta, *Elastic Constants and Elastic Moduli for Metals and Non-metals* (Naukova Dumka, Kiev, 1982) (in Russian).
[24] R. N. Thurston and K. Brugger, Phys. Rev. **133**, 1604 (1964).